

NEW FINITE VOLUME METHOD FOR ROTATING CHANNEL FLOWS INVOLVING BOUNDARY LAYERS

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ABSTRACT. We investigate in this article the boundary layers appearing for a fluid under moderate rotation when the viscosity is small. The fluid is modeled by rotating type Stokes equations known also as the Barotropic mode equations in the primitive equations theory. First we derive the correctors that describe the sharp variations at large Reynolds number (i.e., small viscosity). Second, thanks to a new finite volume method (NFVM) we give numerical solutions of the rotating Stokes system at small viscosity. The NFVM can be applied to a large class of singular perturbation problems.

1. INTRODUCTION

We are interested in this article in the study of boundary layers of a time dependant rotating fluid when the viscosity is small and the boundary is characteristic; this occurs for example when the boundary is solid and at rest. The boundary conditions are considered of homogeneous Dirichlet type. More precisely, we consider the flow in $3D$ verifying the following system:

$$(1.1) \quad \begin{cases} \frac{\partial \mathbf{u}^\varepsilon}{\partial t} - \varepsilon \Delta \mathbf{u}^\varepsilon + \boldsymbol{\omega} \times \mathbf{u}^\varepsilon + \nabla p^\varepsilon = \mathbf{f}, & \text{in } \Omega_\infty \times (0, T), \\ \operatorname{div} \mathbf{u}^\varepsilon = 0, & \text{in } \Omega_\infty \times (0, T), \\ \mathbf{u}^\varepsilon = 0, & \text{on } \partial\Omega_\infty \\ \mathbf{u}^\varepsilon \text{ is } 2\pi\text{-periodic in the } x \text{ and } y \text{ directions,} \\ \mathbf{u}^\varepsilon|_{t=0} = \mathbf{u}_0; \end{cases}$$

see [3] and [8] for more details about the theory of rotating fluids. Here $\boldsymbol{\omega} = \alpha \mathbf{e}_3$ where \mathbf{e}_3 is the unit vector in the canonical basis of \mathbb{R}^3 , $\Omega_\infty = \mathbb{R}^2 \times (0, h)$ is the relevant domain, $\Gamma = \partial\Omega_\infty = \mathbb{R}^2 \times \{0, h\}$ its boundary. The functions \mathbf{u}_0 and \mathbf{f} are given and supposed to be as regular as necessary. Without loss of generality, the constant h will be taken from now equal to 1.

The solution $(\mathbf{u}^\varepsilon, p^\varepsilon)$ of the system (1.1) is such that $\mathbf{u}^\varepsilon(t, x, y, z) = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon) \in \mathbb{R}^3$ and $p^\varepsilon \in \mathbb{R}$, the coefficient ε is a positive constant representing the inverse of the Reynolds number. Throughout this paper the coefficient $\varepsilon > 0$ is intended to be small $\varepsilon \ll 1$. Because of the periodicity conditions (1.1)₄ we will consider a portion of the channel Ω_∞ that we denote by $\Omega = (0, 2\pi) \times (0, 2\pi) \times (0, 1)$ and its boundary $\partial\Omega = (0, 2\pi) \times (0, 2\pi) \times \{0, 1\}$ on which all our calculations will be done. The formal limit solution \mathbf{u}^0 of the system (1.1) is simply obtained

2000 *Mathematics Subject Classification.* 76D10, 76D17, 65L11, 76L05, 68U120.

Key words and phrases. Boundary layer, colocated scheme, correctors, Navier-Stokes equations, finite volumes, singular problems.

by setting $\varepsilon = 0$ in (1.1). Hence, we have:

$$(1.2) \quad \begin{cases} \frac{\partial \mathbf{u}^0}{\partial t} + \boldsymbol{\omega} \times \mathbf{u}^0 + \nabla p^0 = \mathbf{f}, & \text{in } \Omega \times (0, T), \\ \operatorname{div} \mathbf{u}^0 = 0, & \text{in } \Omega \times (0, T), \\ u_3^0 = 0, & \text{on } \partial\Omega, \\ \mathbf{u}^0 \text{ is } 2\pi\text{-periodic in the } x \text{ and } y \text{ directions,} \\ \mathbf{u}^0|_{t=0} = \mathbf{u}_0. \end{cases}$$

The absence in the limit system of the Laplacian term $(-\varepsilon \Delta \mathbf{u}^\varepsilon)$ which is a regularizing term, generates a loss of regularity of the limit solution \mathbf{u}^0 . Thus some discrepancies between the viscous and inviscid solutions appear near the boundary of the domain, that is here $z = 0, 1$. This thin region is called *boundary layer* and the convergence of \mathbf{u}^ε to \mathbf{u}^0 is not expected there especial when we look for the convergence in $H_X^s(\Omega)$ for $s > 1$ and $X = \cdot$. Hence, we introduce a correcting term called *corrector* for which the equation must be of course simpler than the one in the original problem namely (1.1)₁; see [4], [10], [11], [12] and [13] for more details on this notion.

2. THE CORRECTOR EQUATIONS

To study the asymptotic behavior of \mathbf{u}^ε , when $\varepsilon \rightarrow 0$, we propose the following asymptotic expansion of \mathbf{u}^ε :

$$\mathbf{u}^\varepsilon \simeq \mathbf{u}^0 + \boldsymbol{\varphi}^\varepsilon,$$

where $\boldsymbol{\varphi}^\varepsilon$ is the corrector function that will be introduced to correct the difference $\mathbf{u}^\varepsilon - \mathbf{u}^0$ at $z = 0, 1$. The equations verified by $\boldsymbol{\varphi}^\varepsilon$ are as follows:

$$(2.1) \quad \begin{cases} \frac{\partial \boldsymbol{\varphi}^\varepsilon}{\partial t} - \varepsilon \frac{\partial^2 \boldsymbol{\varphi}^\varepsilon}{\partial z^2} + \boldsymbol{\omega} \times \boldsymbol{\varphi}^\varepsilon = 0, & \text{in } \Omega \times (0, T), \\ \operatorname{div} \boldsymbol{\varphi}^\varepsilon = 0, & \text{in } \Omega \times (0, T), \\ \boldsymbol{\varphi}^\varepsilon|_{z=0,1} = -\mathbf{u}^0|_{z=0,1}, \\ \boldsymbol{\varphi}^\varepsilon|_{t=0} = 0. \end{cases}$$

We now introduce an approximate function $\check{\boldsymbol{\varphi}}^\varepsilon$ of $\boldsymbol{\varphi}^\varepsilon$ defined as the sum of $\overline{\boldsymbol{\varphi}}^{0,\varepsilon}$ and $\widetilde{\boldsymbol{\varphi}}^{1,\varepsilon}$ the correctors that we propose to solve the boundary layers at the boundaries $z = 0$ and $z = 1$, respectively:

$$\check{\boldsymbol{\varphi}}^\varepsilon(t, x, y, z) = \overline{\boldsymbol{\varphi}}^{0,\varepsilon}(t, x, y, \frac{z}{\sqrt{\varepsilon}}) + \widetilde{\boldsymbol{\varphi}}^{1,\varepsilon}(t, x, y, \frac{1-z}{\sqrt{\varepsilon}}),$$

where $\bar{z} = \frac{z}{\sqrt{\varepsilon}}$ and $\tilde{z} = \frac{1-z}{\sqrt{\varepsilon}}$. Taking into consideration the linearity of the equations (2.1)_{1,2} and the boundary conditions (2.1)₃, the system verified by $\bar{\varphi}^{0,\varepsilon}$ is given by:

$$(2.2) \quad \begin{cases} \frac{\partial \bar{\varphi}^{0,\varepsilon}}{\partial t} - \frac{\partial^2 \bar{\varphi}^{0,\varepsilon}}{\partial \bar{z}^2} + \boldsymbol{\omega} \times \bar{\varphi}^{0,\varepsilon} = 0, & \text{in } \tilde{\Omega} \times (0, T), \\ \bar{\varphi}^{0,\varepsilon}(\bar{z} = 0) = -\mathbf{u}^0(\bar{z} = 0), \\ \bar{\varphi}^{0,\varepsilon} \rightarrow 0 \text{ as } \bar{z} \rightarrow \infty, \\ \bar{\varphi}^{0,\varepsilon}|_{t=0} = 0. \end{cases}$$

Similarly $\tilde{\varphi}^{1,\varepsilon}$ satisfies the following system:

$$(2.3) \quad \begin{cases} \frac{\partial \tilde{\varphi}^{1,\varepsilon}}{\partial t} - \frac{\partial^2 \tilde{\varphi}^{1,\varepsilon}}{\partial \tilde{z}^2} + \boldsymbol{\omega} \times \tilde{\varphi}^{1,\varepsilon} = 0, & \text{in } \tilde{\Omega} \times (0, T), \\ \tilde{\varphi}^{1,\varepsilon}(\tilde{z} = 0) = -\mathbf{u}^0(\tilde{z} = 0), \\ \tilde{\varphi}^{1,\varepsilon} \rightarrow 0 \text{ as } \tilde{z} \rightarrow \infty, \\ \tilde{\varphi}^{1,\varepsilon}|_{t=0} = 0. \end{cases}$$

Here we denoted by $\tilde{\Omega}$ the stretched domain, i.e., $\tilde{\Omega} = (0, 2\pi) \times (0, 2\pi) \times (0, +\infty)$ of the system (2.2). In the following we will derive the expressions of the solutions of the systems (2.2) and (2.3). For that purpose we need the following proposition.

Proposition 2.1. *Let $\mathbf{u} = \mathbf{u}(t, x, y, z)$ be the solution of the following problem:*

$$(2.4) \quad \begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \frac{\partial^2 \mathbf{u}}{\partial z^2} + \boldsymbol{\omega} \times \mathbf{u} = 0, & \text{in } \tilde{\Omega} \times (0, T), \\ \mathbf{u} = \mathbf{g}, & \text{at } z = 0, \\ \mathbf{u} \rightarrow 0, & \text{as } z \rightarrow +\infty, \\ \mathbf{u} = 0, & \text{at } t = 0. \end{cases}$$

where $\mathbf{g} = (g_1, g_2, 0)$ is a continuous function in $\tilde{\Omega} \times (0, T)$ and $\boldsymbol{\omega} = \alpha \mathbf{e}_3$. Then, the explicit expression of \mathbf{u} is given by:

$$\begin{aligned} \mathbf{u}(t, x, y, z) = & - \int_0^t \frac{\partial K}{\partial z}(t - \tau, z) [(\mathbf{g} - i(\mathbf{e}_3 \times \mathbf{g}))(\tau, x, y, 0) e^{i\alpha(\tau-t)} \\ & + (\mathbf{g} + i(\mathbf{e}_3 \times \mathbf{g}))(\tau, x, y, 0) e^{i\alpha(t-\tau)}] d\tau, \end{aligned}$$

where i is the complex number s.t. $i^2 = -1$, and K is the fundamental solution of the heat equation:

$$K(t, z) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-z^2}{4t}}.$$

Proof . Let $\mathbf{u} = (u_1, u_2, u_3)$ the solution of (2.4). We have $g_3 = 0$, hence $\mathbf{u} = (u_1, u_2, 0)$ i.e. $u_3 = 0$. Taking the cross product of (2.4) with \mathbf{e}_3 , we find:

$$\partial_t(\mathbf{e}_3 \times \mathbf{u}) - \partial_z^2(\mathbf{e}_3 \times \mathbf{u}) - \alpha \mathbf{u} = 0.$$

We then set $\mathbf{C}^\pm = \mathbf{u} \mp i(\mathbf{e}_3 \times \mathbf{u})$, we obtain:

$$\partial_t \mathbf{C}^\pm - \partial_z^2 \mathbf{C}^\pm \pm i\alpha \mathbf{C}^\pm = 0.$$

Denoting by $\mathbf{H}^\pm = \mathbf{C}^\pm e^{\pm i\alpha t}$, one arrives to the following system:

$$(2.5) \quad \begin{cases} \frac{\partial \mathbf{H}^\pm}{\partial t} - \frac{\partial^2 \mathbf{H}^\pm}{\partial z^2} = 0, \text{ in } \tilde{\Omega} \times (0, T), \\ \mathbf{H}^\pm(z=0) = (\mathbf{g}(z=0) \mp i(\mathbf{e}_3 \times \mathbf{g}(z=0)))e^{\pm i\alpha t}, \\ \mathbf{H}^\pm \rightarrow 0, \text{ as } z \rightarrow +\infty, \\ \mathbf{H}^\pm|_{t=0} = 0. \end{cases}$$

Hence \mathbf{H}^\pm satisfies a heat equation with non-homogeneous boundary conditions, then it has the following expression ([2]):

$$\mathbf{H}^\pm = -2 \int_0^t \frac{\partial K}{\partial z}(t-\tau, z) [(\mathbf{g} \mp i(\mathbf{e}_3 \times \mathbf{g}))(\tau, x, y, 0)] e^{\pm i\alpha\tau} d\tau.$$

Then, we infer that:

$$\mathbf{C}^\pm = -2 \int_0^t \frac{\partial K}{\partial z}(t-\tau, z) [(\mathbf{g} \mp i(\mathbf{e}_3 \times \mathbf{g}))(\tau, x, y, 0)] e^{\pm i\alpha(\tau-t)} d\tau.$$

Coming back to \mathbf{u} we have:

$$\mathbf{u} = \frac{1}{2}(\mathbf{C}^+ + \mathbf{C}^-),$$

hence we deduce the explicit expression of the solution of (2.4):

$$\begin{aligned} \mathbf{u} = & - \int_0^t \frac{\partial K}{\partial z}(t-\tau, z) \times \{[(\mathbf{g} - i(\mathbf{e}_3 \times \mathbf{g}))(\tau, x, y, 0)]e^{i\alpha(\tau-t)} + \\ & + [(\mathbf{g} + i(\mathbf{e}_3 \times \mathbf{g}))(\tau, x, y, 0)]e^{i\alpha(t-\tau)}\} d\tau. \end{aligned}$$

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Now, according to Proposition 2.1, the solution of (2.2) $\overline{\varphi}^{0,\varepsilon} = (\overline{\varphi}_1^{0,\varepsilon}, \overline{\varphi}_2^{0,\varepsilon}, \overline{\varphi}_3^{0,\varepsilon})$ has the following expression:

$$\begin{aligned} \overline{\varphi}_j^{0,\varepsilon} = & - \int_0^t \frac{1}{\sqrt{4\pi(t-\tau)}} \frac{z}{2\sqrt{\varepsilon(t-\tau)}} e^{\frac{-z^2}{4\varepsilon(t-\tau)}} \times \{2u_j^0(\tau, x, y, 0) \cos(\alpha(\tau-t)) \\ & + 2(\mathbf{e}_3 \times \mathbf{u}^0)_j(\tau, x, y, 0) \sin(\alpha(\tau-t))\} d\tau, \quad j = 1, 2, \end{aligned}$$

for the two tangential components of $\overline{\varphi}^{0,\varepsilon}$, and the normal component of $\overline{\varphi}^{0,\varepsilon}$ is simply deduced using the incompressibility condition:

$$\begin{aligned} (2.6) \quad \overline{\varphi}_3^{0,\varepsilon} = & - \int_0^t \frac{\sqrt{\varepsilon}}{\sqrt{4\pi(t-\tau)}} e^{\frac{-z^2}{4\varepsilon(t-\tau)}} \times \{-2\partial_z u_3^0(\tau, x, y, 0) \cos(\alpha(\tau-t)) \\ & - 2(\partial_x u_2^0 - \partial_y u_1^0)(\tau, x, y, 0) \sin(\alpha(\tau-t))\} d\tau \\ & + \int_0^t \frac{\sqrt{\varepsilon}}{\sqrt{4\pi(t-\tau)}} e^{\frac{-1}{4\varepsilon(t-\tau)}} \times \{-2\partial_z u_3^0(\tau, x, y, 0) \cos(\alpha(\tau-t)) \\ & - 2(\partial_x u_2^0 - \partial_y u_1^0)(\tau, x, y, 0) \sin(\alpha(\tau-t))\} d\tau. \end{aligned}$$

Then we write the system satisfied by $\overline{\varphi}^{0,\varepsilon}$ which reads as follows:

$$(2.7) \quad \begin{cases} \frac{\partial \overline{\varphi}^{0,\varepsilon}}{\partial t} - \varepsilon \frac{\partial^2 \overline{\varphi}^{0,\varepsilon}}{\partial z^2} + \boldsymbol{\omega} \times \overline{\varphi}^{0,\varepsilon} = (0, 0, \frac{\partial \overline{\varphi}_3^{0,\varepsilon}}{\partial t} - \varepsilon \frac{\partial^2 \overline{\varphi}_3^{0,\varepsilon}}{\partial z^2}), \text{ in } \tilde{\Omega} \times (0, T), \\ \operatorname{div} \overline{\varphi}^{0,\varepsilon} = 0, \text{ in } \tilde{\Omega} \times (0, T), \\ \overline{\varphi}^{0,\varepsilon}(z=0) = (-u_1^0(z=0), -u_2^0(z=0), \overline{\varphi}_3^{0,\varepsilon}(z=0)), \\ \overline{\varphi}^{0,\varepsilon}(z=1) = (\overline{\varphi}_1^{0,\varepsilon}(z=1), \overline{\varphi}_2^{0,\varepsilon}(z=1), 0), \\ \overline{\varphi}^{0,\varepsilon}(t=0) = 0. \end{cases}$$

Now, we need to estimate the right-hand side (denoted hereafter RHS) of (2.7)₁. First we set the change of variables $s = \frac{1}{\sqrt{t-\tau}}$, and we infer that:

$$(2.8) \quad \begin{aligned} \overline{\varphi}_3^{0,\varepsilon} = & - \int_{\frac{1}{\sqrt{t}}}^{\infty} \frac{\sqrt{\varepsilon}}{\sqrt{\pi} s^2} e^{\frac{-z^2 s^2}{4\varepsilon}} \times \left\{ -2\partial_z u_3^0(t - \frac{1}{s^2}, x, y, 0) \cos\left(\frac{\alpha}{s^2}\right) \right. \\ & + 2(\partial_x u_2^0 - \partial_y u_1^0)(t - \frac{1}{s^2}, x, y, 0) \sin\left(\frac{\alpha}{s^2}\right) \left. \right\} ds \\ & + \int_{\frac{1}{\sqrt{t}}}^{\infty} \frac{\sqrt{\varepsilon}}{\sqrt{\pi} s^2} e^{\frac{-s^2}{4\varepsilon}} \times \left\{ -2\partial_z u_3^0(t - \frac{1}{s^2}, x, y, 0) \cos\left(\frac{\alpha}{s^2}\right) \right. \\ & + 2(\partial_x u_2^0 - \partial_y u_1^0)(t - \frac{1}{s^2}, x, y, 0) \sin\left(\frac{\alpha}{s^2}\right) \left. \right\} ds. \end{aligned}$$

By differentiating (2.8) with respect to the time variable t , we obtain:

$$(2.9) \quad \begin{aligned} \frac{\partial \overline{\varphi}_3^{0,\varepsilon}}{\partial t} = & - \frac{\sqrt{\varepsilon}}{\sqrt{t\pi}} e^{\frac{-z^2}{4\varepsilon t}} \times \left\{ -\partial_z u_3^0(0, x, y, 0) \cos(\alpha t) + \right. \\ & + (\partial_x u_2^0 - \partial_y u_1^0)(0, x, y, 0) \sin(\alpha t) \left. \right\} \\ & - \int_{\frac{1}{\sqrt{t}}}^{\infty} \frac{\sqrt{\varepsilon}}{\sqrt{\pi}} \frac{1}{s^2} e^{\frac{-z^2 s^2}{4\varepsilon}} \times \left\{ -2\partial_{tz}^2 u_3^0(t - \frac{1}{s^2}, x, y, 0) \cos\left(\frac{\alpha}{s^2}\right) + \right. \\ & + 2(\partial_{tx}^2 u_2^0 - \partial_{ty}^2 u_1^0)(t - \frac{1}{s^2}, x, y, 0) \sin\left(\frac{\alpha}{s^2}\right) \left. \right\} ds + \\ & - \frac{\sqrt{\varepsilon}}{\sqrt{t\pi}} e^{\frac{-1}{4\varepsilon t}} \times \left\{ -\partial_z u_3^0(0, x, y, 0) \cos(\alpha t) + \right. \\ & + (\partial_x u_2^0 - \partial_y u_1^0)(0, x, y, 0) \sin(\alpha t) \left. \right\} - \\ & - \int_{\frac{1}{\sqrt{t}}}^{\infty} \frac{\sqrt{\varepsilon}}{\sqrt{\pi}} \frac{1}{s^2} e^{\frac{-s^2}{4\varepsilon}} \times \left\{ -2\partial_{tz}^2 u_3^0(t - \frac{1}{s^2}, x, y, 0) \cos\left(\frac{\alpha}{s^2}\right) + \right. \\ & + 2(\partial_{tx}^2 u_2^0 - \partial_{ty}^2 u_1^0)(t - \frac{1}{s^2}, x, y, 0) \sin\left(\frac{\alpha}{s^2}\right) \left. \right\} ds. \end{aligned}$$

We denote by $I_1 + \dots + I_4$ the sum of the terms appearing in the RHS of (2.9). First, we estimate the L^2 -norm of I_1 , we get:

$$\begin{aligned}
 (2.10) \quad \|I_1\|_{L^2(\Omega)}^2 &\leq k\varepsilon \int_0^1 e^{\frac{-z^2}{2\varepsilon t}} dz \\
 &\leq k\varepsilon \int_0^1 e^{\frac{-cz}{\sqrt{\varepsilon t}}} dz, \quad c > 0 \\
 &\leq k\varepsilon^{3/2}.
 \end{aligned}$$

Second, we estimate the L^2 - norm of I_2 , we obtain:

$$\begin{aligned}
 (2.11) \quad \|I_2\|_{L^2(\Omega)}^2 &\leq k \int_0^1 \left(\int_{\frac{1}{\sqrt{t}}}^\infty \frac{\sqrt{\varepsilon}}{\sqrt{4\pi s^2}} e^{\frac{-z^2 s^2}{4\varepsilon}} ds \right)^2 dz \\
 &\leq (\text{using Cauchy Schwartz inequality}) \\
 &\leq k\varepsilon \int_0^1 \int_{\frac{1}{\sqrt{t}}}^\infty \frac{1}{s^2} ds \int_{\frac{1}{\sqrt{t}}}^\infty \frac{1}{s^2} e^{\frac{-z^2 s^2}{2\varepsilon}} ds dz \\
 &\leq k\varepsilon \int_0^1 \int_{\frac{1}{\sqrt{t}}}^\infty \frac{1}{s^2} e^{\frac{-z^2 s^2}{2\varepsilon}} ds dz \\
 &\leq k\varepsilon \int_{\frac{1}{\sqrt{t}}}^\infty \frac{1}{s^2} \int_0^1 e^{\frac{-csz}{\sqrt{\varepsilon}}} dz ds \quad c > 0, \\
 &\leq k\varepsilon^{3/2}.
 \end{aligned}$$

Finally, combining (2.10), (2.11) and the fact that I_3 and I_4 are e.s.t. (where e.s.t. stands for quantities which are exponentially small terms in all $H^m((0, T) \times \Omega)$, $m \geq 0$), we conclude that:

$$(2.12) \quad \left\| \frac{\partial \bar{\varphi}_3^{0,\varepsilon}}{\partial t} \right\|_{L^2(\Omega)} \leq k\varepsilon^{3/4}.$$

We will estimate in the following the z - derivative of $\bar{\varphi}_3^{0,\varepsilon}$ appearing second term in the RHS of (2.7)₁. Hence by differentiating $\bar{\varphi}_3^{0,\varepsilon}$ with respect to the normal variable z , we obtain:

$$\begin{aligned}
 (2.13) \quad \varepsilon \frac{\partial^2 \bar{\varphi}_3^{0,\varepsilon}}{\partial z^2} &= - \int_0^t \frac{\sqrt{\varepsilon}}{\sqrt{4\pi(t-\tau)}} \frac{1}{2(t-\tau)} e^{\frac{-z^2}{4\varepsilon(t-\tau)}} \times \{2\partial_z u_3^0(\tau, x, y, 0) \cos(\alpha(\tau-t)) + \\
 &\quad + 2(\partial_x u_2^0 - \partial_y u_1^0)(\tau, x, y, 0) \sin(\alpha(\tau-t))\} d\tau - \\
 &\quad - \int_0^t \frac{\sqrt{\varepsilon}}{\sqrt{4\pi(t-\tau)}} \frac{z^2}{4(t-\tau)^2} e^{\frac{-z^2}{4\varepsilon(t-\tau)}} \times \{2\partial_z u_3^0(\tau, x, y, 0) \cos(\alpha(\tau-t)) + \\
 &\quad + 2(\partial_x u_2^0 - \partial_y u_1^0)(\tau, x, y, 0) \sin(\alpha(\tau-t))\} d\tau.
 \end{aligned}$$

We denote by $J_1 + J_2$ the sum of the terms in the RHS of (2.13). Multiplying J_1 by z and setting the change of variables $s = \frac{z}{\sqrt{2\varepsilon(t-\tau)}}$, we find:

$$\begin{aligned} zJ_1 &= \frac{1}{\sqrt{2\pi}} \int_{\frac{z}{\sqrt{2\varepsilon t}}}^{\infty} \varepsilon e^{\frac{-s^2}{2}} \times \left\{ -2\partial_z u_3^0\left(t - \frac{z^2}{2\varepsilon s^2}, x, y, 0\right) \cos\left(\frac{\alpha z^2}{2\varepsilon s^2}\right) \right. \\ &\quad \left. + 2(\partial_x u_2^0 - \partial_y u_1^0)\left(t - \frac{z^2}{2\varepsilon s^2}, x, y, 0\right) \sin\left(\frac{\alpha z^2}{2\varepsilon s^2}\right) \right\} ds. \end{aligned}$$

Then, we have

$$\begin{aligned} |zJ_1| &\leq k\varepsilon \int_{\frac{z}{\sqrt{2\varepsilon t}}}^{\infty} e^{\frac{-s^2}{2}} ds \\ &\leq k\varepsilon \int_{\frac{z}{\sqrt{2\varepsilon t}}}^{\infty} e^{-cs} ds, \quad c > 0 \\ &\leq k\varepsilon e^{\frac{-cz}{\sqrt{2\varepsilon t}}}, \end{aligned}$$

and we get the L^2 - norm of the term zJ_1 :

$$\begin{aligned} \|zJ_1\|_{L^2(\Omega)}^2 &\leq k\varepsilon^2 \int_0^1 e^{\frac{-cz}{\sqrt{2\varepsilon t}}} dz \\ &\leq k\varepsilon^{5/2}. \end{aligned}$$

Hence, we obtain

$$(2.14) \quad \|zJ_1\|_{L^2(\Omega)} \leq k\varepsilon^{5/4}.$$

Identically we multiply J_2 by z and apply the same change of variables $s = \frac{z}{\sqrt{\varepsilon(t-\tau)}}$, we get:

$$\begin{aligned} zJ_2 &= \frac{1}{\sqrt{\pi}} \int_{\frac{z}{\sqrt{\varepsilon t}}}^{\infty} 2\varepsilon s^2 e^{\frac{-s^2}{2}} \times \left\{ -\partial_z u_3^0\left(t - \frac{z^2}{2\varepsilon s^2}, x, y, 0\right) \cos\left(\frac{\alpha z^2}{2\varepsilon s^2}\right) \right. \\ &\quad \left. + (\partial_x u_2^0 - \partial_y u_1^0)\left(t - \frac{z^2}{2\varepsilon s^2}, x, y, 0\right) \sin\left(\frac{\alpha z^2}{2\varepsilon s^2}\right) \right\} ds. \end{aligned}$$

Then, we have

$$\begin{aligned} |zJ_2| &\leq \int_{\frac{z}{2\sqrt{\varepsilon t}}}^{\infty} k\varepsilon s^2 e^{\frac{-s^2}{2}} ds \\ &\leq k\varepsilon \left(\frac{z}{2\sqrt{\varepsilon t}} e^{\frac{-z^2}{8\varepsilon t}} + \int_{\frac{z}{2\sqrt{\varepsilon t}}}^{\infty} e^{-s^2/2} ds \right) \\ &\leq k\sqrt{\varepsilon} z e^{\frac{-z^2}{8\varepsilon t}} + 2\varepsilon e^{\frac{-cz}{2\sqrt{\varepsilon t}}}, \end{aligned}$$

and we obtain the L^2 - norm of zJ_2 ,

$$\begin{aligned} \|zJ_2\|_{L^2(\Omega)}^2 &\leq k \int_0^1 \varepsilon z^2 e^{\frac{-z^2}{4\varepsilon t}} dz + k\varepsilon^2 \int_0^1 e^{\frac{-cz}{2\sqrt{\varepsilon t}}} dz \\ &\leq k\varepsilon \int_0^1 z^2 e^{\frac{-cz}{2\sqrt{\varepsilon t}}} dz + k\varepsilon^2 \int_0^1 e^{\frac{-cz}{2\sqrt{\varepsilon t}}} dz \\ &\leq k\varepsilon^{5/2}. \end{aligned}$$

Hence, we infer that

$$(2.15) \quad \|zJ_2\|_{L^2(\Omega)} \leq k\varepsilon^{5/4}.$$

Finally, combining (2.14) and (2.15), we deduce the following estimate:

$$(2.16) \quad \|z\varepsilon \frac{\partial^2 \bar{\varphi}_3^{0,\varepsilon}}{\partial z^2}\|_{L^2(\Omega)} \leq k\varepsilon^{5/4}.$$

3. CONVERGENCE RESULT

In this section we prove the main theoretical result of this article.

Theorem 3.1. *The solution \mathbf{u}^ε of (1.1), with \mathbf{u}_0 and \mathbf{f} supposed to be sufficiently smooth, satisfies the following estimates:*

$$(3.1) \quad \|\mathbf{u}^\varepsilon - \mathbf{u}^0 - \bar{\varphi}^{0,\varepsilon} - \tilde{\varphi}^{1,\varepsilon}\|_{L^\infty(0,T,L^2(\Omega))} \leq k\varepsilon^{3/4},$$

$$(3.2) \quad \|\mathbf{u}^\varepsilon - \mathbf{u}^0 - \bar{\varphi}^{0,\varepsilon} - \tilde{\varphi}^{1,\varepsilon}\|_{L^\infty(0,T,\mathbf{H}^1(\Omega))} \leq k\varepsilon^{1/4},$$

where k is a positive constant depending on the data but not ε , and \mathbf{u}^0 and φ^ε are defined respectively by (1.2) and (2.1). Here we denoted by $\mathbf{L}^2(\Omega) = (L^2(\Omega))^3$ and $\mathbf{H}^1(\Omega) = (H^1(\Omega))^3$.

Proof . First we observe that the corrector φ^ε does not satisfy the desired boundary conditions as given by (2.1)₃, this is due to the choice of a corrector in a simpler form. To overcome this difficulty we introduce additional (small) correctors $\bar{\boldsymbol{\theta}}^\varepsilon$ and $\tilde{\boldsymbol{\theta}}^\varepsilon$ as follows:

$$(3.3) \quad \begin{cases} -\varepsilon \Delta \bar{\boldsymbol{\theta}}^\varepsilon + \nabla \Pi^\varepsilon = 0, & \text{in } \Omega_\infty \times (0, T), \\ \operatorname{div} \bar{\boldsymbol{\theta}}^\varepsilon = 0, \\ \bar{\boldsymbol{\theta}}^\varepsilon|_{z=0} = (0, 0, -\bar{\varphi}_3^{0,\varepsilon}|_{z=0}), \\ \bar{\boldsymbol{\theta}}^\varepsilon|_{z=1} = (-\bar{\varphi}_1^{0,\varepsilon}|_{z=1}, -\bar{\varphi}_2^{0,\varepsilon}|_{z=1}, 0), \end{cases}$$

and

$$(3.4) \quad \begin{cases} -\varepsilon \Delta \tilde{\boldsymbol{\theta}}^\varepsilon + \nabla Q^\varepsilon = 0, & \text{in } \Omega_\infty \times (0, T), \\ \operatorname{div} \tilde{\boldsymbol{\theta}}^\varepsilon = 0, \\ \tilde{\boldsymbol{\theta}}^\varepsilon|_{z=1} = (0, 0, -\tilde{\varphi}_3^{0,\varepsilon}|_{z=1}), \\ \tilde{\boldsymbol{\theta}}^\varepsilon|_{z=0} = (-\tilde{\varphi}_1^{0,\varepsilon}|_{z=0}, -\tilde{\varphi}_2^{0,\varepsilon}|_{z=0}, 0). \end{cases}$$

To estimate the L^2 - norm of the additional correctors, we set $\bar{\boldsymbol{\theta}}^\varepsilon = \sqrt{\varepsilon} \tilde{\boldsymbol{\theta}}^\varepsilon$, $\Pi^\varepsilon = \varepsilon^{3/2} \tilde{\Pi}^\varepsilon$, hence $\tilde{\boldsymbol{\theta}}^\varepsilon$ satisfies the following system:

$$(3.5) \quad \begin{cases} -\Delta \tilde{\boldsymbol{\theta}}^\varepsilon + \nabla \tilde{\Pi}^\varepsilon = 0, & \text{in } \Omega_\infty \times (0, T) \\ \operatorname{div} \tilde{\boldsymbol{\theta}}^\varepsilon = 0, \\ \tilde{\boldsymbol{\theta}}^\varepsilon|_{z=0} = (0, 0, -\frac{\bar{\varphi}_3^{0,\varepsilon}}{\sqrt{\varepsilon}}|_{z=0}), \\ \tilde{\boldsymbol{\theta}}^\varepsilon|_{z=1} = (-\frac{\bar{\varphi}_1^{0,\varepsilon}}{\sqrt{\varepsilon}}|_{z=1}, -\frac{\bar{\varphi}_2^{0,\varepsilon}}{\sqrt{\varepsilon}}|_{z=1}, 0). \end{cases}$$

Then we deduce from the direct estimates of the Stokes problem (see [1]) that:

$$\begin{aligned} \|\tilde{\boldsymbol{\theta}}^\varepsilon\|_{L^2(\Omega)} &\leq k \left\| \frac{\bar{\varphi}_3^{0,\varepsilon}}{\sqrt{\varepsilon}} \right\|_{H^{-1/2}(\Gamma)} + k \left\| \frac{\bar{\varphi}_1^{0,\varepsilon}}{\sqrt{\varepsilon}} \right\|_{H^{-1/2}(\Gamma)} + k \left\| \frac{\bar{\varphi}_2^{0,\varepsilon}}{\sqrt{\varepsilon}} \right\|_{H^{-1/2}(\Gamma)} \\ &\leq k \left\| \frac{\bar{\varphi}_3^{0,\varepsilon}}{\sqrt{\varepsilon}} \right\|_{L^2(\Omega)} + e.s.t. \end{aligned}$$

Now we will estimate the L^2 - norm of $\frac{\overline{\varphi}_3^{0,\varepsilon}}{\sqrt{\varepsilon}}$, hence we have:

$$\begin{aligned}
 \left| \frac{\overline{\varphi}_3^{0,\varepsilon}}{\sqrt{\varepsilon}} \right|^2 &\leq k \left(\int_{\frac{1}{\sqrt{t}}}^{\infty} \frac{1}{4\sqrt{\pi}s^2} e^{\frac{-z^2s^2}{4\varepsilon}} ds \right)^2 \\
 &\leq \text{(Using the Cauchy-Schwartz inequality)} \\
 &\leq k \int_{\frac{1}{\sqrt{t}}}^{\infty} \frac{1}{s^2} ds \int_{\frac{1}{\sqrt{t}}}^{\infty} \frac{1}{s^2} e^{\frac{-z^2s^2}{2\varepsilon}} ds \\
 &\leq k \int_{\frac{1}{\sqrt{t}}}^{\infty} \frac{1}{s^2} e^{\frac{-z^2s^2}{2\varepsilon}} ds.
 \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 \left\| \frac{\overline{\varphi}_3^{0,\varepsilon}}{\sqrt{\varepsilon}} \right\|_{L^2(\Omega)}^2 &\leq k \int_0^1 \int_{\frac{1}{\sqrt{t}}}^{\infty} \frac{1}{s^2} e^{\frac{-z^2s^2}{2\varepsilon}} ds dz \\
 &\leq k \int_{\frac{1}{\sqrt{t}}}^{\infty} \frac{1}{s^2} \int_0^1 e^{\frac{-z^2s^2}{2\varepsilon}} dz ds \\
 &\leq k \int_{\frac{1}{\sqrt{t}}}^{\infty} \frac{1}{s^2} \int_0^1 e^{\frac{-c z s}{\sqrt{\varepsilon}}} dz ds, \quad c > 0 \\
 &\leq k\sqrt{\varepsilon},
 \end{aligned}$$

Hence, we infer that

$$\|\widetilde{\overline{\theta}}^\varepsilon\|_{L^2(\Omega)} \leq k\varepsilon^{1/4}.$$

Finally, we get

$$(3.6) \quad \|\overline{\theta}^\varepsilon\|_{L^2(\Omega)} \leq k\varepsilon^{3/4}.$$

In the following we will estimate the $L^2(\Omega)$ norm of the gradient of $\widetilde{\overline{\theta}}^\varepsilon$, hence we find:

$$\begin{aligned}
 \|\nabla \widetilde{\overline{\theta}}^\varepsilon\|_{L^2(\Omega)} &\leq k \left\| \frac{\overline{\varphi}_3^{0,\varepsilon}}{\sqrt{\varepsilon}} \right|_{z=0} \|_{H^{1/2}(\Gamma)} + k \left\| \frac{\overline{\varphi}_1^{0,\varepsilon}}{\sqrt{\varepsilon}} \right|_{z=1} \|_{H^{1/2}(\Gamma)} + k \left\| \frac{\overline{\varphi}_2^{0,\varepsilon}}{\sqrt{\varepsilon}} \right|_{z=1} \|_{H^{1/2}(\Gamma)} \\
 &\leq k \left\| \frac{\overline{\varphi}_3^{0,\varepsilon}}{\sqrt{\varepsilon}} \right\|_{H^1(\Omega)} + e.s.t \\
 &\leq k\varepsilon^{-1/4}.
 \end{aligned}$$

Thus we deduce that:

$$(3.7) \quad \|\nabla \overline{\theta}^\varepsilon\|_{L^2(\Omega)} \leq k\varepsilon^{1/4}.$$

We notice that the estimate (3.6) also holds for the time derivative of $\overline{\theta}^\varepsilon$, i.e.,

$$(3.8) \quad \left\| \frac{\partial \overline{\theta}^\varepsilon}{\partial t} \right\|_{L^2(\Omega)} \leq k\varepsilon^{3/4}.$$

We now define $w^\varepsilon = u^\varepsilon - u^0 - \overline{\varphi}^{0,\varepsilon} - \widetilde{\varphi}^{1,\varepsilon} - \overline{\theta}^\varepsilon - \widetilde{\theta}^\varepsilon$, and according to (1.1), (1.2), (2.7), (3.3) and (3.4), w^ε verifies:

$$(3.9) \quad \left\{ \begin{array}{l} \frac{\partial \mathbf{w}^\varepsilon}{\partial t} - \varepsilon \Delta \mathbf{w}^\varepsilon + \boldsymbol{\omega} \times \mathbf{w}^\varepsilon + \nabla(p^\varepsilon - p^0 - \Pi^\varepsilon - Q^\varepsilon) = \varepsilon \frac{\partial^2 \bar{\boldsymbol{\varphi}}^{0,\varepsilon}}{\partial x^2} + \varepsilon \frac{\partial^2 \tilde{\boldsymbol{\varphi}}^{1,\varepsilon}}{\partial x^2} \\ + \varepsilon \frac{\partial^2 \bar{\boldsymbol{\varphi}}^{0,\varepsilon}}{\partial y^2} + \varepsilon \frac{\partial^2 \tilde{\boldsymbol{\varphi}}^{1,\varepsilon}}{\partial y^2} + \varepsilon \Delta \mathbf{u}^0 - \boldsymbol{\omega} \times \bar{\boldsymbol{\theta}}^\varepsilon - \boldsymbol{\omega} \times \tilde{\boldsymbol{\theta}}^\varepsilon - \frac{\partial \bar{\boldsymbol{\theta}}^\varepsilon}{\partial t} - \frac{\partial \tilde{\boldsymbol{\theta}}^\varepsilon}{\partial t} \\ + (0, 0, \frac{\partial \bar{\varphi}_3^{0,\varepsilon}}{\partial t}) + (0, 0, \frac{\partial \tilde{\varphi}_3^{1,\varepsilon}}{\partial t}) + (0, 0, \varepsilon \frac{\partial^2 \bar{\varphi}_3^{0,\varepsilon}}{\partial z^2}) + (0, 0, \varepsilon \frac{\partial^2 \tilde{\varphi}_3^{1,\varepsilon}}{\partial z^2}), \quad \text{in } \Omega_\infty \times (0, T), \\ \operatorname{div} \mathbf{w}^\varepsilon = 0, \text{ in } \Omega_\infty \times (0, T), \\ \mathbf{w}^\varepsilon = 0, \quad \text{at } z = 0, 1, \\ \mathbf{w}^\varepsilon \text{ is } 2\pi\text{-periodic in the } x \text{ and } y \text{ directions,} \\ \mathbf{w}^\varepsilon|_{t=0} = 0. \end{array} \right.$$

We multiply (3.9)₁ by \mathbf{w}^ε , integrate over Ω , and apply the Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned} \frac{1}{2} \frac{d\|\mathbf{w}^\varepsilon\|^2}{dt} + \varepsilon \|\nabla \mathbf{w}^\varepsilon\|^2 &\leq \varepsilon \left\| \frac{\partial^2 \bar{\boldsymbol{\varphi}}^{0,\varepsilon}}{\partial x^2} \right\| \|\mathbf{w}^\varepsilon\| + \varepsilon \left\| \frac{\partial^2 \tilde{\boldsymbol{\varphi}}^{1,\varepsilon}}{\partial x^2} \right\| \|\mathbf{w}^\varepsilon\| + \varepsilon \left\| \frac{\partial^2 \bar{\boldsymbol{\varphi}}^{0,\varepsilon}}{\partial y^2} \right\| \|\mathbf{w}^\varepsilon\| + \\ &+ \varepsilon \left\| \frac{\partial^2 \tilde{\boldsymbol{\varphi}}^{1,\varepsilon}}{\partial y^2} \right\| \|\mathbf{w}^\varepsilon\| + \varepsilon \|\Delta \mathbf{u}^0\| \|\mathbf{w}^\varepsilon\| + \|\bar{\boldsymbol{\theta}}^\varepsilon\| \|\mathbf{w}^\varepsilon\| + \|\tilde{\boldsymbol{\theta}}^\varepsilon\| \|\mathbf{w}^\varepsilon\| + \\ &+ \left\| \frac{\partial \bar{\boldsymbol{\theta}}^\varepsilon}{\partial t} \right\| \|\mathbf{w}^\varepsilon\| + \left\| \frac{\partial \tilde{\boldsymbol{\theta}}^\varepsilon}{\partial t} \right\| \|\mathbf{w}^\varepsilon\| + \left\| \varepsilon z \frac{\partial^2 \bar{\varphi}_3^{0,\varepsilon}}{\partial z^2} \right\| \|\nabla \mathbf{w}^\varepsilon\| + \\ &+ \left\| \varepsilon z \frac{\partial^2 \tilde{\varphi}_3^{1,\varepsilon}}{\partial z^2} \right\| \|\nabla \mathbf{w}^\varepsilon\| + \left\| \frac{\partial \bar{\varphi}_3^{1,\varepsilon}}{\partial t} \right\| \|\mathbf{w}^\varepsilon\| + \left\| \frac{\partial \tilde{\varphi}_3^{1,\varepsilon}}{\partial t} \right\| \|\mathbf{w}^\varepsilon\|. \end{aligned}$$

Hence according to (2.12), (2.16), (3.6) and (3.8), we have:

$$\begin{aligned} \frac{1}{2} \frac{d\|\mathbf{w}^\varepsilon\|}{dt} + \varepsilon \|\nabla \mathbf{w}^\varepsilon\|^2 &\leq \frac{1}{2} \|\mathbf{w}^\varepsilon\|^2 + k\varepsilon^{3/2} + k\varepsilon^{3/4} \varepsilon^{1/2} \frac{\|\nabla \mathbf{w}^\varepsilon\|}{2} + k\varepsilon^{3/4} \varepsilon^{1/2} \frac{\|\nabla \mathbf{w}^\varepsilon\|}{2} \\ &\leq \frac{1}{2} \|\mathbf{w}^\varepsilon\|^2 + k\varepsilon^{3/2} + \frac{\varepsilon}{2} \|\nabla \mathbf{w}^\varepsilon\|^2. \end{aligned}$$

In conclusion, we have

$$\frac{d\|\mathbf{w}^\varepsilon\|^2}{dt} + \varepsilon \|\nabla \mathbf{w}^\varepsilon\|^2 \leq \|\mathbf{w}^\varepsilon\|^2 + k\varepsilon^{3/2}.$$

Using the Gronwall inequality, we obtain

$$\|\mathbf{w}^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq k\varepsilon^{3/4} \text{ and } \|\nabla \mathbf{w}^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq k\varepsilon^{1/4}.$$

Hence, according to (3.6), (3.7), and the triangular inequality, we deduce (3.1) and (3.2). This concludes the proof of Theorem 3.1. \bullet

4. A COLLOCATED FINITE VOLUME SCHEME WITH A SPLITTING METHOD FOR THE TIME DISCRETIZATION

We follow here the notations of [7] that we recall in this section for the reader convenience. In the following, we uniformly discretize the domain Ω by using cube finite volumes of dimensions $\Delta x \Delta y \Delta z$:

$$K_{i,j,k} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}] \times [z_{k-\frac{1}{2}}, z_{k+\frac{1}{2}}],$$

where:

$$\begin{aligned} x_{i+\frac{1}{2}} &= i\Delta x, & y_{j+\frac{1}{2}} &= j\Delta y, & z_{k+\frac{1}{2}} &= k\Delta z, \\ \forall i &= 0, \dots, M, \forall j &= 0, \dots, N, \forall k &= 0, \dots, L. \end{aligned}$$

The edges of the control volumes are defined by:

$$\begin{aligned} \Gamma_{i+1/2,j,k} &= \{(x, y, z); x = x_{i+1/2}, y \in [y_{j-1/2}, y_{j+1/2}], z \in [z_{k-1/2}, z_{k+1/2}]\}, \\ \Gamma_{i,j+1/2,k} &= \{(x, y, z); x \in [x_{i-1/2}, x_{i+1/2}], y = y_{j+1/2}, z \in [z_{k-1/2}, z_{k+1/2}]\}, \\ \Gamma_{i,j,k+1/2} &= \{(x, y, z); x \in [x_{i-1/2}, x_{i+1/2}], y \in [y_{j-1/2}, y_{j+1/2}], z = z_{k+1/2}\}, \\ \forall i &= 0, \dots, M, \forall j &= 0, \dots, N, \forall k &= 0, \dots, L. \end{aligned}$$

The velocity and the pressure are approximated in the center of the cells as follows:

$$\begin{aligned} \mathbf{u}_{i,j,k}(t) &\simeq \frac{1}{\Delta x \Delta y \Delta z} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} \mathbf{u}(x, y, z, t) dx dy dz, \\ p_{i,j,k}(t) &\simeq \frac{1}{\Delta x \Delta y \Delta z} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} p(x, y, z, t) dx dy dz. \end{aligned}$$

We also define the velocity fluxes:

$$\begin{aligned} F_{ui+\frac{1}{2},j,k} &\simeq \frac{1}{\Delta y \Delta z} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} u(x_{i+\frac{1}{2}}, y, z, t) dy dz, \\ F_{vi,j+\frac{1}{2},k} &\simeq \frac{1}{\Delta x \Delta z} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} v(x, y_{j+\frac{1}{2}}, z, t) dx dz, \\ F_{wi,j,k+\frac{1}{2}} &\simeq \frac{1}{\Delta x \Delta y} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} w(x, y, z_{k+\frac{1}{2}}, t) dx dy. \end{aligned}$$

4.1. Time discretization. We start by choosing a time discretization for (1.1)₁:

$$(4.1) \quad \frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} - \varepsilon \Delta \mathbf{u}^{n+1} + 2\boldsymbol{\omega} \times \mathbf{u}^n - \boldsymbol{\omega} \times \mathbf{u}^{n-1} + 2\nabla p^n - \nabla p^{n-1} = \mathbf{f}^{n+1}.$$

Thanks to (4.1) we are able to compute the new velocity \mathbf{u}^{n+1} .

Hence, to obtain the pressure, we take the divergence of (1.1)₁ and use the incompressibility condition (1.1)₂ we find:

$$(4.2) \quad \Delta p = \operatorname{div}(\mathbf{f} + \varepsilon \Delta \mathbf{u} + \boldsymbol{\omega} \times \mathbf{u}).$$

Thus we discretize (4.2) as follows:

$$(4.3) \quad \Delta p^{n+1} = \operatorname{div}(\mathbf{f}^{n+1} + \varepsilon \Delta \mathbf{u}^{n+1} - 2\boldsymbol{\omega} \times \mathbf{u}^n - \boldsymbol{\omega} \times \mathbf{u}^{n-1}).$$

By replacing Δ by $-\nabla \times \nabla \times$ (see [7] and [9]), we rewrite (4.3) as below:

$$(4.4) \quad \Delta p^{n+1} = \operatorname{div}(\mathbf{f}^{n+1} - \varepsilon \nabla \times \nabla \times \mathbf{u}^{n+1} - 2\boldsymbol{\omega} \times \mathbf{u}^n - \boldsymbol{\omega} \times \mathbf{u}^{n-1}).$$

Now, by using the relation $\Delta u = \nabla \operatorname{div} \mathbf{u}^{n+1} - \nabla \times \nabla \times \mathbf{u}^{n+1}$, then (4.1) becomes:

$$\begin{aligned} & \mathbf{f}^{n+1} - \varepsilon \nabla \times \nabla \times \mathbf{u}^{n+1} - 2\boldsymbol{\omega} \times \mathbf{u}^n - \boldsymbol{\omega} \times \mathbf{u}^{n-1} \\ &= \frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} - \varepsilon \nabla \operatorname{div} \mathbf{u}^{n+1} + 2\nabla p^n - \nabla p^{n-1}. \end{aligned}$$

Hence, we deduce from (4.4) that

$$(4.5) \quad \Delta p^{n+1} = \operatorname{div}\left(\frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} - \varepsilon \nabla \operatorname{div} \mathbf{u}^{n+1} + 2\nabla p^n - \nabla p^{n-1}\right).$$

Thus, we obtain

$$(4.6) \quad \Delta(p^{n+1} - 2p^n + p^{n-1} + \varepsilon \operatorname{div} \mathbf{u}^{n+1}) = \operatorname{div}\left(\frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t}\right).$$

Then we compute the pressure from

$$(4.7) \quad \begin{cases} \Delta \psi^{n+1} = \operatorname{div}\left(\frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t}\right), \\ \frac{\partial \psi^{n+1}}{\partial n} = 0, \end{cases}$$

and

$$(4.8) \quad p^{n+1} = \psi^{n+1} + 2p^n - p^{n-1} - \varepsilon \operatorname{div} \mathbf{u}^{n+1}.$$

Concerning the boundary conditions, we have the periodicity in the x and y directions and the Dirichlet boundary conditions in the z direction for u^{n+1} :

$$\begin{aligned} \mathbf{u}_{0,j,k}^{n+1} &= \mathbf{u}_{N,j,k}^{n+1}, & \mathbf{u}_{N+1,j,k}^{n+1} &= \mathbf{u}_{1,j,k}^{n+1}, \\ \mathbf{u}_{i,0,k}^{n+1} &= \mathbf{u}_{i,N,k}^{n+1}, & \mathbf{u}_{i,N+1,k}^{n+1} &= \mathbf{u}_{i,1,k}^{n+1}, \\ \frac{\mathbf{u}_{i,j,N+1}^{n+1} + \mathbf{u}_{i,j,N}^{n+1}}{2} &= 0, & \frac{\mathbf{u}_{i,j,0}^{n+1} + \mathbf{u}_{i,j,1}^{n+1}}{2} &= 0. \end{aligned}$$

The Neumann boundary conditions are imposed for ψ^{n+1} in the z direction and the periodicity in x and y directions. Thus, we have

$$\begin{aligned} \psi_{0,j,k}^{n+1} &= \psi_{N,j,k}^{n+1}, & \psi_{N+1,j,k}^{n+1} &= \psi_{1,j,k}^{n+1}, \\ \psi_{i,0,k}^{n+1} &= \psi_{i,N,k}^{n+1}, & \psi_{i,N+1,k}^{n+1} &= \psi_{i,1,k}^{n+1}, \\ \psi_{i,j,N+1}^{n+1} &= \psi_{i,j,N}^{n+1}, & \psi_{i,j,0}^{n+1} &= \psi_{i,j,1}^{n+1}. \end{aligned}$$

The periodicity in x and y for the pressure yields:

$$\begin{aligned} p_{0,j,k} &= p_{M,j,k}, & p_{M+1,j,k} &= p_{1,j,k}, \\ p_{i,0,k} &= p_{i,M,k}, & p_{i,M+1,k} &= p_{i,1,k}, \end{aligned}$$

and for the terms $p_{i,j,0}$ and $p_{i,j,L+1}$ we use the second order compact scheme to compute them:

$$p_{i,j,0} = \frac{5}{2}p_{i,j,1} - 2p_{i,j,2} + \frac{1}{2}p_{i,j,3}, \quad p_{i,j,L+1} = \frac{5}{2}p_{i,j,L} - 2p_{i,j,L-1} + \frac{1}{2}p_{i,j,L-2}.$$

4.2. Finite volume discretization. To Compute the velocity \mathbf{u}^{n+1} , we discretize (4.1) and we obtain:

$$\begin{aligned} & \Delta x \Delta y \Delta z \frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} - \varepsilon \left[\Delta x \Delta y \frac{\mathbf{u}_{i,j,k+1}^{n+1} - 2\mathbf{u}_{i,j,k}^{n+1} + \mathbf{u}_{i,j,k-1}^{n-1}}{\Delta z} \right. \\ & \left. + \Delta y \Delta z \frac{\mathbf{u}_{i+1,j,k}^{n+1} - 2\mathbf{u}_{i,j,k}^{n+1} + \mathbf{u}_{i-1,j,k}^{n+1}}{\Delta x} + \Delta x \Delta z \frac{\mathbf{u}_{i,j+1,k}^{n+1} - 2\mathbf{u}_{i,j,k}^{n+1} + \mathbf{u}_{i,j-1,k}^{n+1}}{\Delta y} \right] \\ & + 2 \left(\begin{array}{c} \frac{\Delta y \Delta z}{2} (p_{i+1,j,k}^n - p_{i-1,j,k}^n) \\ \frac{\Delta x \Delta z}{2} (p_{i,j+1,k}^n - p_{i,j-1,k}^n) \\ \frac{\Delta x \Delta y}{2} (p_{i,j,k+1}^n - p_{i,j,k-1}^n) \end{array} \right) - \left(\begin{array}{c} \frac{\Delta y \Delta z}{2} (p_{i+1,j,k}^{n-1} - p_{i-1,j,k}^{n-1}) \\ \frac{\Delta x \Delta z}{2} (p_{i,j+1,k}^{n-1} - p_{i,j-1,k}^{n-1}) \\ \frac{\Delta x \Delta y}{2} (p_{i,j,k+1}^{n-1} - p_{i,j,k-1}^{n-1}) \end{array} \right) \\ & + \Delta x \Delta y \Delta z (\boldsymbol{\omega} \times (2\mathbf{u}_{i,j,k}^n - \mathbf{u}_{i,j,k}^{n-1})) = \Delta x \Delta y \Delta z \mathbf{f}_{i,j,k}^{n+1}. \end{aligned}$$

To compute the pressure we first compute ψ^{n+1} :

$$\begin{aligned} & \Delta x \Delta y \frac{\psi_{i,j,k+1}^{n+1} - 2\psi_{i,j,k}^{n+1} + \psi_{i,j,k-1}^{n+1}}{\Delta z} + \Delta y \Delta z \frac{\psi_{i+1,j,k}^{n+1} - 2\psi_{i,j,k}^{n+1} + \psi_{i-1,j,k}^{n+1}}{\Delta x} \\ & + \Delta x \Delta z \frac{\psi_{i,j+1,k}^{n+1} - 2\psi_{i,j,k}^{n+1} + \psi_{i,j-1,k}^{n+1}}{\Delta y} = \frac{1}{2\Delta t} [\Delta y \Delta z [(3F_{ui+\frac{1}{2}jk}^{n+1} - 4F_{ui+\frac{1}{2}jk}^n + F_{ui+\frac{1}{2}jk}^{n-1}) \\ & - (3F_{ui-\frac{1}{2}jk}^{n+1} - 4F_{ui-\frac{1}{2}jk}^n + F_{ui-\frac{1}{2}jk}^{n-1})] + \Delta x \Delta z [(3F_{vij+\frac{1}{2}k}^{n+1} - 4F_{vij+\frac{1}{2}k}^n + F_{vij+\frac{1}{2}k}^{n-1}) \\ & - (3F_{vij-\frac{1}{2}k}^{n+1} - 4F_{vij-\frac{1}{2}k}^n + F_{vij-\frac{1}{2}k}^{n-1})] + \Delta x \Delta y [(3F_{wijk+\frac{1}{2}}^{n+1} - 4F_{wijk+\frac{1}{2}}^n + F_{wijk+\frac{1}{2}}^{n-1}) \\ & - (3F_{wijk-\frac{1}{2}}^{n+1} - 4F_{wijk-\frac{1}{2}}^n + F_{wijk-\frac{1}{2}}^{n-1})]]. \end{aligned}$$

Then, we easily obtain the pressure:

$$\begin{aligned} p_{i,j,k}^{n+1} &= \psi_{i,j,k}^{n+1} + 2p_{i,j,k}^n - p_{i,j,k}^{n-1} - \frac{\varepsilon}{\Delta x \Delta y \Delta z} [\Delta y \Delta z (F_{ui+\frac{1}{2}jk}^{n+1} - F_{ui-\frac{1}{2}jk}^{n+1}) \\ & + \Delta x \Delta z (F_{vij+\frac{1}{2}k}^{n+1} - F_{vij-\frac{1}{2}k}^{n+1}) + \Delta x \Delta y (F_{wijk+\frac{1}{2}}^{n+1} - F_{wijk-\frac{1}{2}}^{n+1})]. \end{aligned}$$

4.3. Computation of the fluxes. We recall here that the simplest method to compute the fluxes (linear interpolation) does not work when the viscosity ε is small. Hence the authors in [7] considered a modified interpolation method for the fluxes in two dimensional case. Now, since we aim here to study the boundary layers at small viscosity, we need, on the one hand, to adapt the discretization in [7] to the 3D dimensional case and, on the other hand, to introduce the correctors in the finite volume discretization basis that is the NFVM. Thus we first start by introducing the 3D fluxes inherited from [7]:

$$\begin{aligned} F_{u_{i+\frac{1}{2},j,k}^{n+1}} &= \frac{u_{i+1,j,k}^{n+1} + u_{i,j,k}^{n+1}}{2} + \theta \frac{\Delta y \Delta z}{4a} (p_{i+2,j,k}^n - 2p_{i+1,j,k}^n + p_{i,j,k}^n) \\ &\quad - \theta \frac{\Delta y \Delta z}{4a} (p_{i+1,j,k}^n - 2p_{i,j,k}^n + p_{i-1,j,k}^n), \end{aligned}$$

$$\begin{aligned}
F_{v_{i+\frac{1}{2},j,k}^{n+1}} &= \frac{v_{i,j+1,k}^{n+1} + v_{i,j,k}^{n+1}}{2} + \theta \frac{\Delta x \Delta z}{4a} (p_{i,j+2,k}^n - 2p_{i,j+1,k}^n + p_{i,j,k}^n) \\
&\quad - \theta \frac{\Delta x \Delta z}{4a} (p_{i,j+1,k}^n - 2p_{i,j,k}^n + p_{i,j-1,k}^n), \\
F_{w_{i,j,k+\frac{1}{2}}^{n+1}} &= \frac{w_{i,j,k+1}^{n+1} + w_{i,j,k}^{n+1}}{2} + \theta \frac{\Delta x \Delta y}{4a} (p_{i,j,k+2}^n - 2p_{i,j,k+1}^n + p_{i,j,k}^n) \\
&\quad - \theta \frac{\Delta x \Delta y}{4a} (p_{i,j,k+1}^n - 2p_{i,j,k}^n + p_{i,j,k-1}^n), \\
&\quad \forall i = 0, \dots, M, \forall j = 0, \dots, N, \forall k = 0, \dots, L,
\end{aligned}$$

where: θ is the relaxation coefficient and

$$a = \frac{3\Delta x \Delta y \Delta z}{2\Delta t} + 2\varepsilon \frac{\Delta x \Delta y}{\Delta z} + 2\varepsilon \frac{\Delta y \Delta z}{\Delta x} + 2\varepsilon \frac{\Delta x \Delta z}{\Delta y}.$$

5. NEW FINITE VOLUME DISCRETIZATION

In this section we introduce a new finite volume schemes, that is we approximate the solution of (1.1) by:

$$\mathbf{u}_h = \sum_{i,j=1} \mathbf{r}_{i,j,0} \hat{\varphi}_{i,j,0}^{0,\varepsilon} \chi_{i,j,0} + \sum_{i,j=1} \mathbf{r}_{i,j,L+1} \hat{\varphi}_{i,j,L+1}^{1,\varepsilon} \chi_{i,j,L+1} + \sum_{i,j,1} \mathbf{u}_{i,j,k} \chi_{i,j,k},$$

where:

$$\begin{aligned}
h &= \Delta z, \\
\mathbf{r}_{i,j,0} &= \frac{\mathbf{u}_{i,j,0} + \mathbf{u}_{i,j,1}}{2}, \\
\mathbf{r}_{i,j,L+1} &= \frac{\mathbf{u}_{i,j,L+1} + \mathbf{u}_{i,j,L}}{2}, \\
\chi_{i,j,0} &= \chi_{(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}) \times (0, h)}, \\
\chi_{i,j,N+1} &= \chi_{(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}) \times ((L-1)h, Lh)}, \\
\chi_{i,j,k} &= \chi_{(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}) \times (z_{k-\frac{1}{2}}, z_{k+\frac{1}{2}})},
\end{aligned}$$

and

$$\begin{aligned}
\hat{\varphi}_i^{0,\varepsilon} &= - \int_0^t \frac{1}{\sqrt{4\pi(t-\tau)}} \frac{z}{2\sqrt{\varepsilon}(t-\tau)} e^{\frac{-z^2}{4\varepsilon(t-\tau)}} \\
&\quad \times \{2\tau \cos(\alpha(\tau-t)) - 2\tau \sin(\alpha(\tau-t))\} d\tau, \quad \forall i = 1, 2, \\
\hat{\varphi}_i^{1,\varepsilon} &= - \int_0^t \frac{1}{\sqrt{4\pi(t-\tau)}} \frac{z}{2\sqrt{\varepsilon}(t-\tau)} e^{\frac{-(1-z^2)}{4\varepsilon(t-\tau)}} \\
&\quad \times \{2\tau \cos(\alpha(\tau-t)) - 2\tau \sin(\alpha(\tau-t))\} d\tau, \quad \forall i = 1, 2. \\
\hat{\varphi}_3^{0,\varepsilon} &= \hat{\varphi}_3^{1,\varepsilon} = 0.
\end{aligned}$$

Multiplying (1.1)₁ by $\chi_{i,j,k}$, integrating over Ω , and replacing \mathbf{u}^ε by \mathbf{u}_h we find that the equations are the same as the classical finite volume scheme (4.9). Moreover the correctors verify (2.2)₁. Hence they do not contribute to these equations. For the numerical simulations

we do not use the modified boundary layer $\overline{\varphi}^{0,\varepsilon}$ and $\tilde{\varphi}^{1,\varepsilon}$ directly. Instead we consider another approximate form which reads as follows:

$$\tilde{\tilde{\varphi}}^{0,\varepsilon}(t, z) = (-\exp(\frac{-z^2}{4\varepsilon t}), -\exp(\frac{-z^2}{4\varepsilon t}), 0).$$

Indeed, The approximation $\tilde{\tilde{\varphi}}^{0,\varepsilon}$ is much easier to be implemented numerically in coding than the theoretical corrector $\overline{\varphi}^{0,\varepsilon}$ obtained in section 2.

Due to the nodes $\mathbf{r}_{i,j,0}$ and $\mathbf{r}_{i,j,L+1}$, the linear system associated with this scheme is not closed. However, by adding the correctors, we ensuring the closure of the linear system corresponding to the NFVM considered. Hence, We multiply (4.1) by the corrector $\tilde{\varphi}^{0,\varepsilon}$ and integrate over $K_{i,j,1}$, we find:

$$(5.1) \int_{K_{ij1}} \frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} \tilde{\varphi}^{0,\varepsilon} - \varepsilon \int_{K_{ij1}} \Delta \mathbf{u}^{n+1} \tilde{\varphi}^{0,\varepsilon} + \int_{K_{ij1}} \boldsymbol{\omega} \times (2\mathbf{u}^n - \mathbf{u}^{n-1}) \tilde{\varphi}^{0,\varepsilon} \\ + 2 \int_{K_{ij1}} \nabla p^n \tilde{\varphi}^{0,\varepsilon} - \int_{K_{ij1}} \nabla p^{n-1} \tilde{\varphi}^{0,\varepsilon} = \int_{K_{ij1}} \mathbf{f}^{n+1} \tilde{\varphi}^{0,\varepsilon}.$$

In the following we calculate each term of (5.1), for the first term in the LHS (left-hand side) of (5.1) we find:

$$\int_{K_{ij1}} \frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} \tilde{\varphi}^{0,\varepsilon} dx dy dz = \frac{3\mathbf{u}_{i,j,1}^{n+1} - 4\mathbf{u}_{i,j,1}^n + \mathbf{u}_{i,j,1}^{n-1}}{2\Delta t} \int_{K_{ij1}} \tilde{\varphi}^{0,\varepsilon} dx dy dz.$$

For the second term in the LHS of (5.1), we obtain:

$$(5.2) \int_{K_{ij1}} \Delta \mathbf{u}^{n+1} \tilde{\varphi}^{0,\varepsilon} dx dy dz = - \int_{K_{ij1}} \nabla \mathbf{u}^{n+1} \nabla \tilde{\varphi}^{0,\varepsilon} dx dy dz + \int_{\partial K_{ij1}} \tilde{\varphi}^{0,\varepsilon} \frac{\partial \mathbf{u}^{n+1}}{\partial n} d\Gamma, \\ = - \int_{K_{ij1}} \frac{\partial \mathbf{u}^{n+1}}{\partial z} \frac{\partial \tilde{\varphi}^{0,\varepsilon}}{\partial z} dx dy dz + \int_{\partial K_{ij1}} \tilde{\varphi}^{0,\varepsilon} \frac{\partial \mathbf{u}^{n+1}}{\partial n} d\Gamma.$$

Now, we calculate the first term in the RHS of (5.2) we find:

$$\int_{K_{ij1}} \nabla \mathbf{u}^{n+1} \nabla \tilde{\varphi}^{0,\varepsilon} dx dy dz = \int_{K_{ij1}} \frac{\partial \mathbf{u}^{n+1}}{\partial z} \frac{\partial \tilde{\varphi}^{0,\varepsilon}}{\partial z} dx dy dz \\ = \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{i-1/2}}^{y_{i+1/2}} \int_0^{h/2} \frac{\partial \mathbf{u}^{n+1}}{\partial z} \frac{\partial \tilde{\varphi}^{0,\varepsilon}}{\partial z} dx dy dz \\ + \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{i-1/2}}^{y_{i+1/2}} \int_{h/2}^h \frac{\partial \mathbf{u}^{n+1}}{\partial z} \frac{\partial \tilde{\varphi}^{0,\varepsilon}}{\partial z} dx dy dz \\ = \frac{\mathbf{u}_{i,j,1}^{n+1} - \mathbf{r}_{i,j,1}^{n+1}}{h/2} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{i-1/2}}^{y_{i+1/2}} \int_0^{h/2} \frac{\partial \tilde{\varphi}^{0,\varepsilon}}{\partial z} dx dy dz \\ + \frac{\mathbf{u}_{i,j,2}^{n+1} - \mathbf{u}_{i,j,1}^{n+1}}{h} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{i-1/2}}^{y_{i+1/2}} \int_{h/2}^h \frac{\partial \tilde{\varphi}^{0,\varepsilon}}{\partial z} dx dy dz \\ = \frac{2}{h} (\mathbf{u}_{ij1}^{n+1} - \mathbf{r}_{ij0}^{n+1}) \Delta x \Delta y (\tilde{\varphi}^{0,\varepsilon}(\frac{h}{2}) - \tilde{\varphi}^{0,\varepsilon}(0)) \\ + \frac{\mathbf{u}_{i,j,2}^{n+1} - \mathbf{u}_{i,j,1}^{n+1}}{h} \Delta x \Delta y (\tilde{\varphi}^{0,\varepsilon}(h) - \tilde{\varphi}^{0,\varepsilon}(h/2)).$$

For the second term in the RHS of (5.2) we obtain:

$$\begin{aligned}
\int_{\partial K_{ij1}} \tilde{\varphi}^{0,\varepsilon} \frac{\partial \mathbf{u}^{n+1}}{\partial n} d\Gamma &= \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{i-1/2}}^{y_{i+1/2}} \int_{z=0} \tilde{\varphi}^{0,\varepsilon} \left(-\frac{\partial \mathbf{u}}{\partial z}\right) d\Gamma \\
&+ \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{i-1/2}}^{y_{i+1/2}} \int_{z=h} \tilde{\varphi}^{0,\varepsilon} \left(\frac{\partial \mathbf{u}}{\partial z}\right) d\Gamma \\
&+ \int_{x_{i-1/2}}^{x_{i+1/2}} \int_0^h \int_{y=y_{j-1/2}} \tilde{\varphi}^{0,\varepsilon} \left(-\frac{\partial \mathbf{u}}{\partial y}\right) d\Gamma \\
&+ \int_{x_{i-1/2}}^{x_{i+1/2}} \int_0^h \int_{y=y_{j+1/2}} \tilde{\varphi}^{0,\varepsilon} \left(\frac{\partial \mathbf{u}}{\partial y}\right) d\Gamma \\
&+ \int_{y_{i-1/2}}^{y_{i+1/2}} \int_0^h \int_{x=x_{i-1/2}} \tilde{\varphi}^{0,\varepsilon} \left(-\frac{\partial \mathbf{u}}{\partial x}\right) d\Gamma \\
&+ \int_{y_{i-1/2}}^{y_{i+1/2}} \int_0^h \int_{x=x_{i+1/2}} \tilde{\varphi}^{0,\varepsilon} \left(\frac{\partial \mathbf{u}}{\partial x}\right) d\Gamma.
\end{aligned}$$

Now, the third term in the LHS of (5.1), can be rewritten as bellow:

$$\int_{K_{ij1}} \boldsymbol{\omega} \times (2\mathbf{u}^n - \mathbf{u}^{n-1}) \tilde{\varphi}^{0,\varepsilon} dx dy dz = \boldsymbol{\omega} \times (2\mathbf{u}_{i,j,1}^n - \mathbf{u}_{i,j,1}^{n-1}) \Delta x \Delta y \int_0^h \tilde{\varphi}^{0,\varepsilon} dx dy dz.$$

We calculate the first component of the fourth term in the LHS of (5.1) and, we find:

$$\int_{K_{ij1}} \partial_x p^{n+1} \tilde{\varphi}_1^{0,\varepsilon} dx dy dz = \frac{p_{i+1,j,1}^{n+1} - p_{i-1,j,1}^{n+1}}{2\Delta x} \Delta x \Delta y \int_0^h \tilde{\varphi}_1^{0,\varepsilon} dz.$$

For the second component of the fourth term in the LHS of (5.1) we have:

$$\int_{K_{ij1}} \partial_y p^{n+1} \tilde{\varphi}_2^{0,\varepsilon} dx dy dz = \frac{p_{i,j+1,1}^{n+1} - p_{i,j-1,1}^{n+1}}{2\Delta y} \Delta x \Delta y \int_0^h \tilde{\varphi}_2^{0,\varepsilon} dz.$$

Concerning the first term on the RHS of (5.1), we have:

$$\int_{K_{ij1}} \mathbf{f}^{n+1} \tilde{\varphi}^{0,\varepsilon} dx dy dz = \Delta x \Delta y \mathbf{f}_{i,j,1}^{n+1} \int_0^h \tilde{\varphi}^{0,\varepsilon} dz.$$

Hence, we infer that

$$\begin{aligned}
& \frac{1}{2\Delta t}(3\mathbf{u}_{i,j,1}^{n+1} - 4\mathbf{u}_{i,j,1}^n + \mathbf{u}_{i,j,1}^{n-1}) \int_0^h \widetilde{\varphi}^{0,\varepsilon} dz - \varepsilon \left[\frac{1}{h}(-3\widetilde{\varphi}^{0,\varepsilon}(h/2)\mathbf{u}_{i,j,1}^{n+1} + \right. \\
& + 2\mathbf{r}_{i,j,0}^{n+1}\widetilde{\varphi}^{0,\varepsilon}(h/2) + \mathbf{u}_{i,j,2}^{n+1}\widetilde{\varphi}^{0,\varepsilon}(h/2)) - \left(\frac{1}{(\Delta x)^2}(\mathbf{u}_{i-1,j,1}^{n+1} - 2\mathbf{u}_{i,j,1}^{n+1} + \mathbf{u}_{i+1,j,1}^{n+1}) + \right. \\
& + \frac{1}{(\Delta y)^2}(\mathbf{u}_{i,j-1,1}^{n+1} - 2\mathbf{u}_{i,j,1}^{n+1} + \mathbf{u}_{i,j+1,1}^{n+1})) \int_0^h \widetilde{\varphi}^{0,\varepsilon} dz] + \boldsymbol{\omega} \times (2\mathbf{u}_{i,j,1}^n - \mathbf{u}_{i,j,1}^{n-1}) \int_0^h \widetilde{\varphi}^{0,\varepsilon} dz + \\
& + 2 \begin{pmatrix} \left(\frac{p_{i+1,j,1}^n - p_{i-1,j,1}^n}{2\Delta x} \right) \int_0^h \widetilde{\varphi}^{0,\varepsilon} dz \\ \left(\frac{p_{i,j+1,1}^n - p_{i,j-1,1}^n}{2\Delta y} \right) \int_0^h \widetilde{\varphi}^{0,\varepsilon} dz \\ 0 \end{pmatrix} - \begin{pmatrix} \left(\frac{p_{i+1,j,1}^{n-1} - p_{i-1,j,1}^{n-1}}{2\Delta x} \right) \int_0^h \widetilde{\varphi}^{0,\varepsilon} dz \\ \left(\frac{p_{i,j+1,1}^{n-1} - p_{i,j-1,1}^{n-1}}{2\Delta y} \right) \int_0^h \widetilde{\varphi}^{0,\varepsilon} dz \\ 0 \end{pmatrix} \\
& = \mathbf{f}_{i,j,1}^{n+1} \int_0^h \widetilde{\varphi}^{0,\varepsilon} dz.
\end{aligned}$$

6. NUMERICAL RESULTS

In this section we will compute the error approximation using the classical finite volume method and the new finite volume method, so the pressure and the source term are chosen such that:

$$p(x, y, z, t) = \cos(2\pi x) \cos(2\pi y) \cos(\pi z) t,$$

$$u^\varepsilon(x, y, z, t) = t \sin(2\pi y) \left(1 - e^{\frac{-z}{\sqrt{\varepsilon}}} \cos\left(\frac{z}{\sqrt{\varepsilon}}\right)\right) \left(1 - e^{\frac{-(1-z)}{\sqrt{\varepsilon}}} \cos\left(\frac{(1-z)}{\sqrt{\varepsilon}}\right)\right),$$

$$v^\varepsilon(x, y, z, t) = t \sin(2\pi x) \left(1 - e^{\frac{-z}{\sqrt{\varepsilon}}} \cos\left(\frac{z}{\sqrt{\varepsilon}}\right)\right) \left(1 - e^{\frac{-(1-z)}{\sqrt{\varepsilon}}} \cos\left(\frac{(1-z)}{\sqrt{\varepsilon}}\right)\right),$$

and

$$w^\varepsilon(x, y, z, t) = 0.$$

Note that the test solution given above satisfies the equations (1.1)_{1,2} including the boundary and initial conditions (1.1)_{3,5} with $\mathbf{u}_0 = 0$.

Thus, the function source is chosen using this test solution.

N=M=L	t	ε	CFVM	NFVM
10	1	10^{-2}	0.03206	0.12836
20	1	10^{-2}	0.00634	0.03893
30	1	10^{-2}	0.00269	0.02553
10	1	10^{-3}	0.092294	0.22647
20	1	10^{-3}	0.033726	0.15753
30	1	10^{-3}	0.01331	0.08020
10	1	10^{-5}	1.61660e+03	0.04487
20	1	10^{-5}	0.08741	0.010303
30	1	10^{-5}	0.11722	0.00460
10	1	10^{-6}	1.10612e+10	0.044901
20	1	10^{-6}	4.42881e+06	0.01032
30	1	10^{-6}	1.12960e+03	0.00442
10	1	10^{-7}	5.26218e+62	0.04490
20	1	10^{-7}	1.16428e+29	0.01032
30	1	10^{-7}	6.72495e+17	0.00443

FIGURE 1. The L^2 norm of the velocity error with classical finite volume (CFVM) and new finite volume method (NFVM) for different values of ε at $t = 1$

N=M=L	t	ε	CFVM	NFVM
10	1	10^{-2}	0.02493	0.03178
20	1	10^{-2}	0.00511	0.00920
30	1	10^{-2}	0.00224	0.00533
10	1	10^{-3}	0.02684	0.02771
20	1	10^{-3}	0.00553	0.00907
30	1	10^{-3}	0.002381	0.00590
10	1	10^{-5}	1.48996e+02	0.02602
20	1	10^{-5}	0.00774	0.00539
30	1	10^{-5}	0.00655	0.00238
10	1	10^{-6}	1.01953e+16	0.026016
20	1	10^{-6}	2.83861e+05	0.00539
30	1	10^{-6}	58.98117	0.00238
10	1	10^{-7}	4.85027e+61	0.02601
20	1	10^{-7}	7.46273e+27	0.005394
30	1	10^{-7}	3.51186e+16	0.00238

FIGURE 2. The L^2 norm of the pressure error with classical finite volume (CFVM) and new finite volume method (NFVM) for different values of ε at $t = 1$

7. CONCLUSION AND FRACTURE WORKS

In this paper we have compared two different finite volume methods (CFVM) and (NFVM) when the viscosity is considered small and more precisely in the rang $10^{-3} - 10^{-7}$. We derived an approximate solution of the-time dependent rotating fluid in 3D channel using the splitting methods for the time discretization and colocated space discretization. One of the novelties of this article is that we propose a new numerical approach to treat the pressure and the divergence free condition introducing correctors to solve the boundary layers. We also show that the (NFVM) is more performing than the (CFVM) when the viscosity is small. We showed that our NFVM still perform for very large Reynolds number. To the best of our knowledge, this is the first work which gives a new finite volume scheme taking into account boundary layer variations for the linearized Navier-Stokes equations. Note that the method developed here may apply to many other problems and domains. This will be the subject of subsequent work.

8. APPENDIX.

In this paragraph, we give a sketch of the proof of the existence and regularity of the solution of the limit problem (1.2). For complete study of the existence of system (1.2) we refer the reader to , see also and .We first want to apply the Hille-Phillips-Yosida Theorem to prove the existence and uniqueness of the solution of (1.2). Thus we start by introducing the adequate function spaces:

$$H = \{ \mathbf{v} \in (L^2(\Omega))^3; \operatorname{div} \mathbf{v} = 0, v_3(z=0) = v_3(z=h) = 0, \\ \text{and } \mathbf{v} \text{ is } 2\pi \text{ periodic in the } x \text{ and } y \text{ directions} \}.$$

$$D(A) = \{ \mathbf{v} \in H; \exists p \in \mathbf{D}'(\Omega), \text{ such } \boldsymbol{\omega} \times \mathbf{v} + \nabla p \in H \},$$

with the norm:

$$(8.1) \quad \|\mathbf{v}\|_{D(A)} = (\|\mathbf{v}\|_H^2 + \|\boldsymbol{\omega} \times \mathbf{v} + \nabla p\|_H^2)^{1/2}.$$

Then for $\mathbf{v} \in D(A)$ we set $A\mathbf{v} = \boldsymbol{\omega} \times \mathbf{v} + \nabla p$, thus we define an unbounded linear operator A which maps $D(A) \subset H$ onto H .

Theorem 8.1 (Hille-Yosida Theorem). *Let H be a Hilbert space and let $B : D(B) \rightarrow H$ a linear unbounded operator, with domain $D(B) \subset H$ such that $D(B)$ is dense in H and $(-B)$ is m -dissipative. Then $(-B)$ is the infinitesimal generator of a contraction semigroup $\{S(t)\}_{t>0}$ in H , and the solution of the following system:*

$$(8.2) \quad \begin{cases} \frac{d\mathbf{v}}{dt} + B\mathbf{v} = \mathbf{f}, \\ \mathbf{v}|_{t=0} = \mathbf{v}_0, \end{cases}$$

satisfies the following properties:

(H₀) *If \mathbf{v}_0 and $\mathbf{f} \in L^1(0, T; H)$, then $\mathbf{v} \in C([0, T]; H)$, $\forall T > 0$.*

(H₁) *If $\mathbf{v}_0 \in D(B)$ and $\mathbf{f}' \in L^1(0, T; H)$ then $\mathbf{v} \in C^1([0, T]; H) \cap C^0([0, T]; D(B))$ and $\frac{d\mathbf{v}}{dt} \in L^\infty([0, T]; H)$, $\forall T > 0$.*

Remark 1. *A linear operator is dissipative in H if and only if: $\forall \mathbf{u} \in D(A), \forall \lambda > 0, \|\mathbf{u} - \lambda A\mathbf{u}\| \geq \|\mathbf{u}\|$.*

Remark 2. A linear operator A is m -dissipative if: A is dissipative and $\forall f \in X, \forall \lambda > 0, \exists u \in D(A), u - \lambda Au = f$.

Proof . Now we want to show that the operator $(-A)$ is m -dissipative, hence we will prove that the following system:

$$(8.3) \quad \begin{cases} \lambda \omega \times u + \lambda \nabla p + u = f, \\ \operatorname{div} u = 0, \\ u_3 = 0, \text{ en } z = 0, 1, \end{cases}$$

has a unique solution in $D(A)$ for all $f \in H$ and $\forall \lambda > 0$, and the solution satisfies the estimate:

$$(8.4) \quad \|u\|_H \leq \|f\|_H, \quad \forall f \in H.$$

We multiply (8.3) by $v \in H$, integrate over Ω and we find:

$$\lambda \int_{\Omega} (\omega \times u) \cdot v d\Omega + \lambda \int_{\Omega} \nabla p \cdot v d\Omega + \int_{\Omega} u \cdot v d\Omega = \int_{\Omega} f v d\Omega.$$

We have:

$$\int_{\Omega} \nabla p v = - \int_{\Omega} p \operatorname{div} v + \int_{\partial\Omega} p v \cdot n d(\Gamma) = 0.$$

We set

$$a(u, v) = \lambda \int_{\Omega} (\omega \times u) \cdot v + \int_{\Omega} u \cdot v,$$

and

$$F(v) = \int_{\Omega} f v.$$

Here $a(., .)$ is a continuous and coercive bilinear form in $H \times H$. In fact we have:

$$\begin{aligned} |a(u, v)| &\leq \lambda |u|_H |v|_H + |u|_H |v|_H, \\ &\leq k(\lambda) |u|_H |v|_H, \end{aligned}$$

and

$$|a(u, u)| = |u|_H^2.$$

Also $F(v)$ is a continuous linear form:

$$\int_{\Omega} f v \leq \|f\| |v|.$$

Hence according to the Lax-Milligram theorem, there exists a unique $u \in H$ such that:

$$\lambda Au + u = f,$$

that is,

$$\lambda \omega \times u + \lambda \nabla p + u = f.$$

Multiplying the above equation by u and integrating over Ω , we find:

$$\lambda \int_{\Omega} (\omega \times u) u + \lambda \int_{\Omega} \nabla p u + \int_{\Omega} u u = \int_{\Omega} f u,$$

then the solution u satisfies the estimate:

$$\|u\|_H \leq \|f\|_H.$$

Also we have:

$$\begin{aligned}\|\mathbf{u}\|_{D(A)} &= (\|\mathbf{u}\|_H^2 + \|\boldsymbol{\omega} \times \mathbf{u} + \nabla p\|_H^2)^{1/2}, \\ &\leq \|\mathbf{u}\|_H + \|\boldsymbol{\omega} \times \mathbf{u} + \nabla p\|_H, \\ &\leq k(\lambda)\|\mathbf{f}\|_H.\end{aligned}$$

Hence $(-A)$ is m-dissipative operator, Moreover we have $\mathbf{u}_0 \in H$, then according to the Hille-Yosida theorem the system (1.2) has a unique solution $\mathbf{u} \in C([0, \infty[, H)$. Furthermore, we have:

$$\begin{aligned}\|\nabla p\|_{H^{-1}} &\leq \frac{1}{\lambda}\|\mathbf{f}\|_{H^{-1}} + \frac{1}{\lambda}\|\mathbf{u}\|_{H^{-1}} + \|\boldsymbol{\omega} \times \mathbf{u}\|_{H^{-1}} \\ &\leq k(\lambda)\|\mathbf{f}\|_H.\end{aligned}$$

Then, we obtain;

$$\|p\|_{L^2(\Omega)} \leq k(\lambda)\|\mathbf{f}\|_H.$$

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9. ACKNOWLEDGEMENTS.

We are very grateful to Sylvain Faure for his collaboration to programming the code Matlab in section 4.

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